Serial Number 32

THE COLLEGES OF OXFORD UNIVERSITY

Entrance Examination in Mathematics

MATHEMATICS I

13 November 1995 Afternoon

Time allowed: 3 hours

Answers to each of Sections A, B and C must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B or C.

All candidates must attempt Question 1 which carries twice the mark for any other question. There is no restriction on the number of questions any candidate may attempt but only Question 1 and the best three solutions to Questions 2-11 will contribute to the total mark for this paper.

The use of calculators is allowed, but, unless otherwise stated, exact answers should be given.

SECTION A

A1. (i) Express $\dfrac{2a}{x^2 - a^2}$ in partial fractions.

(ii) Let $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$. Find a real number t and a vector $\mathbf{p}$, perpendicular to $\mathbf{a}$, such that $\mathbf{b} = t\mathbf{a} + \mathbf{p}$.

(iii) Find all real values of x such that $2|x| + |x - 3| \geq 6$.

(iv) Evaluate $\displaystyle\sum_{r=1}^{n}(-1)^r r$ when n is even.

(v) Find the sums $\displaystyle\sum_{r=1}^{\infty} x^r$ and $\displaystyle\sum_{r=1}^{\infty} rx^r$, where $|x| < 1$.

(vi) Differentiate $\dfrac{e^{\cos x}}{\cos(\sin x)}$ with respect to x.

(vii) Evaluate $\int_1^2 x \ln(x^2)\, dx$.

(viii) Find the greatest and least values of $e^{-x}(x^2 - 4x + 4)$ when $0 \leq x \leq 3$.

(ix) Find the area of the finite region enclosed by the curves $y = x^2 - 2$ and $y = x$.

(x) Find all real values of x and y such that $\cos x + \cos y = 0$ and $x + y = \pi/4$.

SECTION B

B2. Find all solutions of the equations

$$\begin{array}{rcrcrcl} x & + & & & (1-\lambda)z & = & 3 \\ 2x & + & \lambda y & + & z & = & 8 \\ -x & + & 3\lambda y & + & (2+\lambda)z & = & 3 \end{array}$$

(a) when $\lambda = 0$, (b) when $\lambda = 1$ and (c) when $\lambda = 2$.

B3. Let α, β and γ be the solutions of the cubic equation $x^3 + Ax^2 + Bx + C = 0$.

(a) Show that

$$\begin{aligned} \alpha + \beta + \gamma &= -A \\ \beta\gamma + \gamma\alpha + \alpha\beta &= B \\ \alpha\beta\gamma &= -C. \end{aligned}$$

(b) In terms of A, B and C find $\alpha^2 + \beta^2 + \gamma^2$ and $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$.

(c) When $A = 4$, $B = 8$ and $C = 16$ find the cubic equation the solutions of which are α^2, β^2 and γ^2.

B4. Let z be the complex number $\alpha + i\beta$, where $\beta \neq 0$.

(a) Find the quadratic equation $x^2 + bx + c = 0$ with b and c real which has z as a solution. What is the other solution?

(b) Show that if z is a solution of the equation $p(x) = 0$, where $p(x)$ is a polynomial with real coefficients, then $\bar{z} = \alpha - i\beta$ is also a solution of $p(x) = 0$.

(c) Deduce that, for any polynomial $p(x)$ of odd degree with real coefficients, the equation $p(x) = 0$ has at least one real solution.

B5. Let p and q be integers with $p > 0$ and $q > 0$ and let $d > 0$ be the smallest integer which can be written in the form

$$d = \alpha p + \beta q,$$

where α and β are integers.

(a) Let c be an integer such that both p and q are integer multiples of c. Show that d is an integer multiple of c.

(b) Show that $d \leq p$.

(c) Show that integers λ and r can be found such that $p = \lambda d + r$, with $\lambda \geq 0$ and $0 \leq r \leq d - 1$.

(d) Show that r can be written in the form

$$r = \alpha' p + \beta' q$$

for suitable integers α' and β'.

(e) Deduce that $r = 0$ and that p is an integer multiple of d.

(f) Show that if p and q are prime numbers with $p \neq q$ then $d = 1$.

B6. For each real number θ let R_θ be the matrix defined by

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Show that, for real numbers θ and ϕ, $R_\theta R_\phi = R_{(\theta+\phi)}$.

(b) Show that, for each positive integer n, $(R_\theta)^n = R_{n\theta}$.

(c) Deduce that $(R_\theta)^n = I$ if and only if, for some integer m, $\theta = \dfrac{2m\pi}{n}$.

(d) When $0 \leq \theta < 2\pi$ find all values of ϕ such that $(R_\phi)^n = R_\theta$.

SECTION C

C7. Let

$$f(x) = x^3 + ax^2 + x + 1,$$

where a is a real number. Determine the values of a for which the function f has

(a) two turning points,

(b) a stationary point of inflection,

(c) no turning points and no stationary point of inflection.

Find the greatest value and the least value of $f(x)$ for $0 \leq x \leq 1$ (i) when $a \geq 0$, and (ii) when $a = -2$.

C8. Using the substitution $y = \pi - x$, or otherwise, show that

$$\int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx.$$

Evaluate the integral

$$\int_0^{\pi} x \sin^3 x\, dx.$$

C9. (a) Find and classify any turning points and describe the behaviour as x approaches $+\infty$ and $-\infty$ of the function

$$y = 9x^2 e^{-x}.$$

Sketch the graph of the function.

(b) Show that if $0 < a < 36e^{-2}$ then the equation $9x^2 = ae^x$ has one negative solution and two positive solutions.

(c) Let c be the larger positive solution of the equation $9x^2 = e^x$. Find the largest integer n such that $n \leq c$.

C10. The functions $f(x)$ and $g(x)$ satisfy the equations

$$\frac{d}{dx}\left(p(x)\frac{df}{dx}\right) = \lambda f(x), \qquad \frac{d}{dx}\left(p(x)\frac{dg}{dx}\right) = \mu g(x),$$

where λ and μ are constants.

(a) Using integration by parts, show that, if $f(a) = f(b) = g(a) = g(b) = 0$, then

$$\int_a^b \left(p(x)\frac{df}{dx}\right)\frac{dg}{dx}\,dx = -\lambda \int_a^b f(x)g(x)\,dx = -\mu \int_a^b f(x)g(x)\,dx.$$

(b) Deduce that, if $\lambda \neq \mu$, then

$$\int_a^b f(x)g(x)\,dx = 0.$$

(c) Deduce that, for positive integers m and n, with $m \neq n$,

$$\int_0^\pi \sin mx \sin nx\,dx = 0.$$

C11. (a) Solve the differential equation

$$\frac{dy}{dx} = \frac{1-y^2}{x} \qquad (x > 0)$$

for y in terms of x, given that $y = 0$ when $x = 1$.

(b) Show that, as x approaches 0, y approaches -1. What happens to y as x approaches infinity?

(c) Solve the same differential equation, but with the condition that $y = 3$ when $x = 1$.

(d) Sketch the graphs of both solutions.